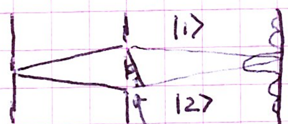
95-615 Quantum Mechanics I Meg Noah

1 The Necessity of Quantum Mechanics

1. Classical Mechanics doesn't explain atoms.
   Why are they stable? An accelerating (orbiting) e^- should radiate/collapse.
   If you assume harmonics $\nu \propto \left(\frac{1}{n^2} - \frac{1}{m^2}\right)$ each normal mode corresponds to a degree of freedom $\Rightarrow$ dE/dT (specific heat) would be huge!
2. Photoelectric effect
   $E = h\nu$ $\quad p = \frac{E}{c}$ $\quad \lambda = \frac{h}{p}$ $\quad$ e^- diffraction by crystal = wave
   particle ↗ ↖ wave/light
3. Distance btwn atoms
4. Polarization $|\alpha\rangle = c_1|\parallel\rangle + c_2|\perp\rangle$ $\quad c_1, c_2$ are real $\quad c_1^2$ = fraction passes filter
   Malus' Law $\cos^2\theta$ $\quad$ light = stream of photons $\quad c_2^2$ = not pass filter

> In any measurement, there is some disturbance
> In QM it can't be neglected

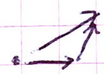
4. Interference (Young's Experiment)
   $I = 4I_0$ (bright fringe)
   $I = 0$ (dark fringe)



$d\sin\theta = m\lambda$

Probability of finding
$P = |\psi(x)|^2 = |\psi_1 + \psi_2|^2$
$= |\psi_1|^2 + |\psi_2|^2 + \underbrace{\psi_1^*\psi_2 + \psi_1\psi_2^*}_{\text{interference term}}$

Mathematical Scheme to Account for Superposition
Linear Vector Space - analogous to arrows (direction + length)
- general
- infinite dimensions
- defined by a number of axioms


$\vec{A} + \vec{B} = \vec{C}$

Notation
$|\,\rangle$ - ket vector
$|A\rangle$ - specific vector
$|A\rangle + |B\rangle = |C\rangle$ in same vector space linear combination $\quad c_1|A\rangle + c_2|B\rangle = c_3|C\rangle$ (can be complex)

$|0\rangle$: null vector
$|A\rangle + |0\rangle = |A\rangle$: identity of addition
$|A\rangle + |-A\rangle = |0\rangle$: for every vector there is an additive inverse

Proof that null vector is unique:
$|A\rangle + |0'\rangle = |A\rangle$
N/w show that $|0'\rangle \equiv |0\rangle$

Concept of Orthogonality:
when the inner product is $\emptyset$

> Complex Linear Vector Space
> $$\sum_i^N c_i|i\rangle = |0\rangle$$
> If all the $|i\rangle$'s are linearly independent, then $c_i = 0$ for all i.
>
> If not $c_i = 0$ for all i then $|i\rangle$ are not linearly independent.

② Proof: $|0\rangle + |0'\rangle = 0|V\rangle + 0'|V\rangle = (0+0')|V\rangle$

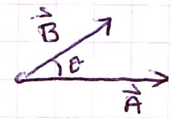
Classical: 3 position, 3 momentum specifies state, know state @ time t_1
use Newton's laws to get state at time t_2

Q Mech: Can't know position + momentum simultaneously
State of system is a vector in a complex linear vector space
of infinite dimensions known as a Hilbert space

Linear Vector Spaces

maximum number of independent vectors = 3 in 3-space

$\vec{A} \rightarrow |A\rangle$ $\quad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = \sum_{i=1}^{3} A_i B_i$



Scalar Product $\langle A|B\rangle = \langle B|A\rangle^*$ $\quad \langle A|A\rangle = \langle A|A\rangle^* \geq 0$ ($=0$ only if $|A\rangle = |0\rangle$)

Length of Vector $\sqrt{\langle A|A\rangle}$

linear: $\langle A|R\rangle = c_1\langle A|B\rangle + c_2\langle A|C\rangle \leftarrow$ given $|R\rangle = c_1|B\rangle + c_2|C\rangle$

antilinear: $\langle R|A\rangle = \langle A|R\rangle^* = c_1^*\langle B|A\rangle + c_2^*\langle C|A\rangle$

$|A\rangle$ "ket" linear space of ket vectors
$\langle A|$ "bra" one corresponds for every ket
$a|A\rangle \rightarrow \langle A|a^*$

Linearly independent vectors $|V\rangle = \sum_{i=1}^{n} v_i|i\rangle$
the components v_i (coef of vector)
are unique.

Proof: Suppose $|V\rangle = \sum v_i'|i\rangle$ then $|V\rangle - |V\rangle = |0\rangle = \sum(v_i - v_i')|i\rangle$
$\therefore v_i = v_i'$

$|-V\rangle = -|V\rangle$
$|0\rangle = 0|V\rangle$
$\langle V|W\rangle = 0 \Rightarrow |V\rangle$ and $|W\rangle$ are orthogonal.

Gram-Schmidt Procedure (to get orthonormal vectors)

$|I\rangle, |II\rangle, |III\rangle, \ldots$

$|1\rangle = \dfrac{|I\rangle}{\sqrt{\langle I|I\rangle}}$ $\quad \langle 1|1\rangle = \dfrac{\langle I|I\rangle}{\langle I|I\rangle} = 1$

$|2'\rangle = |II\rangle - \langle 1|II\rangle|1\rangle$ subtract projection of II onto I

$\langle 1|2'\rangle = \langle 1|II\rangle - \langle 1|II\rangle\langle 1|1\rangle = 0 \Rightarrow |2\rangle = \dfrac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} \Rightarrow \langle 2|2\rangle = \dfrac{\langle 2'|2'\rangle}{\langle 2'|2'\rangle} = 1$ mutual orthog

$|3'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle$
$\langle 1|3'\rangle = \langle 2|3'\rangle = 0$
$|3\rangle = \dfrac{|3'\rangle}{\sqrt{\langle 3'|3'\rangle}}$

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Vector Products - Great Advantages of Dirac Notation!

$|v\rangle = \sum_i v_i |i\rangle \qquad |w\rangle = \sum_j w_j |j\rangle$

$\langle v|w\rangle = \sum_{j=1}^{n} v_j^* \langle j| \sum_{i=1}^{n} w_i |i\rangle = \sum_i \sum_j v_i^* w_j \underbrace{\langle j|i\rangle}_{\delta_{ij}} = \sum_i \sum_j v_i^* w_j \delta_{ij} = \sum_i v_i^* w_i$

given $\langle w|v\rangle = \sum_{i=1}^{n} w_i^* v_i$

can easily find v_i

$|v\rangle = \sum_i v_i |i\rangle = \sum |i\rangle v_i = \sum |i\rangle\langle i|v\rangle \quad \langle j|v\rangle = \sum v_i \langle j|i\rangle = v_j \delta_{ij} = v_j$

$\therefore \; v_i = \langle i|V\rangle$

Identity Operator $\underbrace{\sum |i\rangle\langle i|}_{I} V\rangle \qquad |w\rangle = \sum |i\rangle \underbrace{\langle i|w\rangle}_{\text{component } w \text{ along } i \text{ basis}}$

Regard Basis as (once you chose a basis)

ket as column vector $|V\rangle \rightarrow \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ bra as row vector $\langle w| = (w_1^* \; w_2^* \cdots w_n^*)$

Scalar Product: $\langle w|V\rangle = (w_1^* \; w_2^* \cdots w_n^*) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = w_1^* v_1 + w_2^* v_2 + \cdots w_n^* v_n = \sum_{i=1}^{n} w_i^* v_i$

QM Linear Operators

State is represented by ket vector

Observables are represented by a particular set of linear operators

$\Omega|v\rangle = |v'\rangle$

* must satisfy linear operator $\hat{\Omega}\{c_1|V_1\rangle + c_2|V_2\rangle\} = c_1 \hat{\Omega}|V_1\rangle + c_2 \hat{\Omega}|V_2\rangle$

Adjoint of a linear operator $|v\rangle \rightarrow \langle V| \quad c|V\rangle \rightarrow \langle V|c^* \quad \Rightarrow \boxed{\begin{array}{l} \hat{\Omega}|v\rangle = |v'\rangle \\ \langle v'| = \langle V|\Omega^+ \end{array}}$ adjoint Ω^+

Hermetian: Eigenvalue = Real, Eigenvector = orthogonal

$|v\rangle = \sum v_i |i\rangle$

$\Omega|v\rangle = \sum_i v_i \Omega|i\rangle = \sum v_i |i'\rangle = |v'\rangle$

$\langle j|v'\rangle = \sum_i v_i \langle j|i'\rangle = \sum_i v_i \langle j|\hat{\Omega}|i\rangle = \sum_i v_i \Omega_{ji} \Rightarrow v_j' = \sum_i \Omega_{ji} v_i$

Linear Operators $|V'\rangle = \Omega|V\rangle$ square matrix:

$$|V\rangle = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} \quad |V'\rangle = \begin{pmatrix} V_1' \\ V_2' \\ \vdots \\ V_n' \end{pmatrix} \quad \begin{pmatrix} V_1' \\ V_2' \\ \vdots \\ V_n' \end{pmatrix} = \begin{vmatrix} \langle 1|\Omega|1\rangle & \langle 1|\Omega|2\rangle & \cdots & \langle 1|\Omega|n\rangle \\ \langle 2|\Omega|1\rangle & \langle 2|\Omega|2\rangle & \cdots & \langle 2|\Omega|n\rangle \\ \vdots & \vdots & & \vdots \\ \langle n|\Omega|1\rangle & \langle n|\Omega|2\rangle & & \langle n|\Omega|n\rangle \end{vmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}$$

Ω is adjoint

$\Omega|i\rangle = |i'\rangle \Leftrightarrow \langle i'| = \langle i|\Omega^+$

Can think of as the element of Ω^+ matrix: $\langle i'|j\rangle = \langle i|\Omega^+|j\rangle$
$= \langle j|i'\rangle^* = \langle j|\Omega|i\rangle^*$

$\Omega^+_{ij} = \Omega^*_{ji}$ the complex conjugate of the transpose of Ω

$\Omega = \Omega^+$ = Hermitean = self-adjoint (can also have $\Omega = -\Omega^+$ anti Herm)

Very important property of Hermetian op: $\Omega|V\rangle = |V'\rangle$
$\Omega|W\rangle = \omega'|W'\rangle$

eigenvector eigenvalue

In general $\Omega|w_i'\rangle = w_i'|w_i'\rangle$ n-independent eigenvalues

Theorem: If Ω is Hermetian, then the eigenvalues w_i are real numbers.

Also. The eigenvectors belonging to different eigenvalues are orthogonal

Proof: $\Omega|w_i'\rangle = w_i'|w_i'\rangle$

$\Omega|w_j'\rangle = w_j'|w_j'\rangle \Rightarrow \langle w_j'|\Omega^+ = \langle w_j^*|w_j'^*$

(A) $\langle w_j'|\Omega^+|w_i'\rangle = w_j'^*\langle w_j'|w_i'\rangle$

(B) $\langle w_j'|\Omega|w_i'\rangle = w_i'\langle w_j'|w_i'\rangle$

(A)−(B)=0 because $\Omega^+ = \Omega \Rightarrow 0 = (w_j'^* - w_i)\langle w_j'|w_i'\rangle$ $i=j$?

If $i=j$ $0 = (w_i'^* - w_i')\underbrace{\langle w_i'|w_i'\rangle}_{=1}$ $\therefore w_i'^* = w_i'$ $\therefore$ real.

Also: If $i \neq j$ $\underbrace{(w_j'^* - w_i')}_{\text{non-zero generally}}\langle w_j|w_i\rangle = 0$ $\therefore$ orthogonal

Unitary Operator: $\Omega^+ = \Omega^{-1}$ where $\Omega^{-1}\Omega = \mathbb{I}$ identity operator
$\mathbb{I}|V\rangle = |V\rangle$

the eigenvalues are not real
complex numbers of magnitude 1

$\Omega|w_i'\rangle = w_i'|w_i'\rangle$

Note: To get the inverse matrix, cofactor transpose divide by determinant.

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(3) Observable = Hermetian op.
State = ket vector

eigenvalues are real, can only predict probabilities - how to find?

$\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle$

$(\Omega - \omega_i \mathbb{I})|\omega_i\rangle = |0\rangle$

$|\omega_i\rangle = (\Omega - \omega_i \mathbb{I})^{-1}|0\rangle$

$|i\rangle\langle i|V\rangle = \langle i|V\rangle|i\rangle$ = projection operator

$\sum_i |i\rangle\langle i| = \mathbb{I}$ = identity operator Completeness

$\langle j|\omega_i\rangle = \langle j|\underbrace{(\Omega - \omega_i \mathbb{I})^{-1}}_{\text{matrix}}|0\rangle = \sum_l \langle j|(\Omega - \omega_i \mathbb{I})^{-1}|l\rangle\langle l|0\rangle$ 2

Read appendix A-1

M^{-1} = Transpose cofactor of M = det M $\therefore$ $\det(\Omega - \omega_i \mathbb{I}) = 0$ $n \times n$ matrix
polynomial of degree n is solved with n roots:

$\omega_i^n + c_1\omega_i^{n-1} + c_2\omega^{n-2} + \dots + c_0 = 0$

the roots of the 'circular' equation are the eigenvalues
once you get the eigenvalues, put back into eq. to get eigenvectors.

Unitary Operator $\quad U|u_i\rangle = u_i|u_i\rangle$

$\langle u_i|U^\dagger = \langle u_i|u_i^*$

$\langle u_i|U^\dagger U|u_j\rangle = u_i^* u_j \langle u_i|u_j\rangle = \langle u_i|u_j\rangle \Rightarrow (1 - u_i^* u_j)\langle u_i|u_j\rangle = 0$

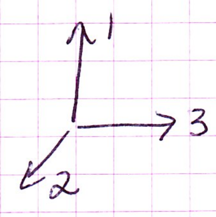
if $u_i \neq u_j$ then $\langle u_i|u_j\rangle = 0$

if $i = j$ then $(1 - u_i^* u_i)\underbrace{\langle u_i|u_i\rangle}_{\neq 0} = 0 \quad \therefore \quad u_i^* u_i = 1$ assume $|u_i\rangle \neq |0\rangle$

that means $|u_i|^2 = 1$ u_i is a complex unit vector (mag = 1)

example: $u_i = e^{i\alpha}$

Example of Unitary Operator: Rotation



1 = x
2 = y
3 = z

$R(\frac{\pi}{2}\hat{i})$ $\quad 2 \to 3$, $3 \to -2$

$i \to |1\rangle \quad R(\frac{\pi}{2}\hat{i})|1\rangle = |1\rangle$

$j \to |2\rangle \quad R(\frac{\pi}{2}\hat{i})|2\rangle = |3\rangle$

$k \to |3\rangle \quad R(\frac{\pi}{2}\hat{i})|3\rangle = -|2\rangle$

$$R(\tfrac{\pi}{2}\hat{i}) = \begin{vmatrix} \langle 1|R|1\rangle & \langle 1|R|2\rangle & \langle 1|R|3\rangle \\ \langle 2|R|1\rangle & \langle 2|R|2\rangle & \langle 2|R|3\rangle \\ \langle 3|R|1\rangle & \langle 3|R|2\rangle & \langle 3|R|3\rangle \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

the matrix in this particular representation or basis

Calculate the eigenvalues

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \lambda\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1, \pm i$$

get eigenvectors
can only get to a constant phase factor
$c_1 = e^{i\alpha}$

$c_1 = ic_1 \quad -c_3 = ic_2$

$X_{+1} = \begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad X_{+i} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ -i \end{pmatrix} \quad ic_2 = c_3$

$X_{+1}^* X_{+1} = (c_1^* \; 0 \; 0)\begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix} = 1$

$c_1^2 = 1$ no change if $c_1 = e^{i\alpha}$

NOTE BOOK

Finding eigenvalues + eigenvectors (non-diag values are non-zero)

$\langle \omega_i | \Omega | \omega_j \rangle = \omega_j \langle \omega_i | \omega_j \rangle = \omega_j \delta_{ij} \omega_i$

If you chose the eigenvectors as the basis, the eigenvalues in the diagonal are the matrix of the operator.

$|\omega_i\rangle = U|i\rangle$

$\langle i | U^\dagger = \langle \omega_i |$ $\quad |\omega_j\rangle = U|j\rangle \quad \langle i|j\rangle = \langle \omega_i|\omega_j\rangle = \delta_{ij} = \langle i|U^\dagger U|j\rangle$

Can find Unitary Matrix via: $|\omega_j\rangle = U|j\rangle \quad \langle i|\omega_j\rangle = \langle i|U|j\rangle$

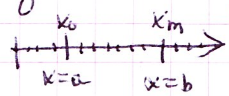
ith element of ω_i col. vector.

$$U = \begin{array}{ccc} \text{column 1} = \text{eigenvector 1} & C2 = C2 & C3 = C3 \\ \left[\;\;\right] & \left[\;\;\right] & \left[\;\;\right] \end{array}$$

$\langle i | U^\dagger \Omega U | j \rangle$ diagonal operator

whenever you write a matrix, that means you chose a bases. there are infinite numbers of basis: question of convergence

$\boxed{\text{Function Space}}$



$f(x) \quad a \leq x \leq b$

$f(x_i)$ can be considered as a component of a vector in $(n+1)$ space

$g(x_i)$ another vector in $(n+1)$ space

$$\langle V|W\rangle = \sum_i v_i^* w_i \Leftrightarrow \langle f|g\rangle = \sum_{i=0}^{n} f^*(x_i) g(x_i) \underbrace{(x_{i+1} - x_i)}_{\Delta x_i} \quad \lim_{n\to 0} \rightarrow \int_a^b f^*(x) g(x)\, dx$$

$\Delta x_i \leftarrow$ infinitesimal gap

Completeness: $\sum_x |x\rangle\langle x| \Delta x = \int |x\rangle\langle x|\, dx = \mathbb{I}$

$$\langle f|g\rangle = \langle f|\mathbb{I}|g\rangle = \int_a^b \langle f|x\rangle\langle x|g\rangle\, dx$$
$$= \int_a^b f^*(x) g(x)\, dx$$

$\langle x|g\rangle = g(x)$ component of g projected on x

$\langle x|f\rangle = f(x)$

$\langle f|x\rangle = \langle x|f\rangle^* = f^*(x)$

Coordinate (x) representation of a ket vector

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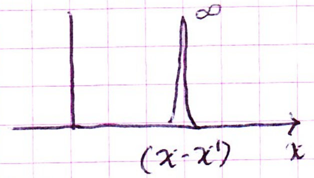
(4) Function Space and the Dirac Function $\boxed{\langle x|f\rangle = f(x)}$

$$\langle f|g\rangle = \int \langle f|x\rangle\langle x|g\rangle dx = \int \langle x|f\rangle^* g(x)\,dx = \int f^*(x)\, g(x)\,dx$$

$\cancel{\langle x|g\rangle}$ ᴗ $\langle x|x'\rangle = \delta(x-x') \Rightarrow$ derivative of the dirac delta function

$$\lim_{\Delta\to 0} g_\Delta(x) = \left(\frac{1}{\pi\Delta^2}\right)^{1/2} e^{-x^2/\Delta^2} = \delta(x)$$



$$\delta'(x-x') = \lim_{\Delta\to 0} \frac{d}{dx} g_\Delta(x-x')$$

for any finite Δ you can take the derivative then let the limit go to zero

if g is even g' is odd for any finite Δ $\boxed{\delta'(x-x') = \frac{d}{dx}\delta(x-x')}$

What are the properties of Dirac

$f(x) = \int f(x')\,\delta(x-x')\,dx'$ assuming the integral limits/range include the point $x=x'$

$$\frac{df}{dx} = \frac{d}{dx}\int f(x')\,\delta(x-x')\,dx' = \int f(x')\,\delta'(x-x')\,dx'$$

$\delta(x-x') = \delta(x'-x)$ property: delta is even + real

$\delta'(x-x') = -\delta'(x-x')$ property: delta prime is odd

$\boxed{\text{Fourier Transform}}$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int g(k)\, e^{ikx}\, dk$$

$$g(k) = \frac{1}{\sqrt{2\pi}}\int f(x)\, e^{-ikx}\, dx$$

The D Operator

Definition $\langle x|D|f\rangle \equiv \frac{df}{dx} = \int dx' \langle x|D|x'\rangle\langle x'|f\rangle$

$$= \int dx'\, \delta'(x-x')\, f(x')$$

$$= \int dx' \langle x|D|x'\rangle\, f(x')$$

Also Define as: $\langle x|D|x'\rangle \equiv \delta'(x-x')$

Is D Hermetian or self-adjoint? (equal to transpose/conjugate?)

$\langle x|D|x'\rangle = \langle x|D^\dagger|x'\rangle \stackrel{?}{=} \langle x'|D|x\rangle^*$ if $\Omega_{ij} = \Omega_{ji}^*$ it is Hermitian

$$\left.\begin{aligned}\langle x|D|x'\rangle &= \delta'(x-x') \\ \langle x'|D|x\rangle^* &= [\delta'(x'-x)]^* = \delta'(x'-x) = -\delta'(x-x')\end{aligned}\right\} \therefore \text{ no, it is anti-hermitian}$$

The K Operator

To make D Hermitean, multiply by i ($\sqrt{-1}$): $K = -iD$

$$\langle x'|K|x\rangle = \langle x|K|x'\rangle^*$$

K operator continued ...

This should be true with any arbitrary ket vector f, g

$\langle f|K|g\rangle \overset{?}{=} \langle f|K^{+}|g\rangle \overset{?}{=} \langle g|K|f\rangle^{*}$

$$\langle f|K|g\rangle = \iint dx\,dx' \langle f|x\rangle\langle x|K|x'\rangle\langle x'|g\rangle$$
$$= \iint dx\,dx'\, f^{*}(x)\left[-i\delta'(x-x')\,g(x')\right.$$
$$= -i\int dx\, f^{*}(x)\int dx'\, g(x')\,\delta(x-x')$$
$$= -i\int dx\, f^{*}(x)\frac{dg(x)}{dx}$$

Is the RHS the same? $\quad g = \langle x|g\rangle$

$$\langle g|K|f\rangle^{*} = \left[\iint dx\,dx' \langle g|x\rangle\langle x|K|x'\rangle\langle x'|f\rangle\right]^{*} \quad g^{*}(x)(-i\delta'(x-x'))f(x')$$

do the x' integration first: (by parts)

$$= \left[\int dx\, g^{*}(x)\int dx'\,(-i\delta'(x-x'))\,f(x')\right]^{*}$$
$$= +i\int_a^b dx\, g(x)\frac{df^{*}(x)}{dx} = \underbrace{i\,g(x)\,f^{*}(x)\Big|_a^b}_{\to 0} - i\int f^{*}(x)\frac{dg}{dx}\,dx$$
$$= -i\int dx\, f^{*}(x)\frac{dg(x)}{dx}$$

$\therefore \langle f|K|g\rangle = \langle g|K|f\rangle^{*}$ if $\quad i\,g(x)\,f^{*}(x)\Big|_a^b \to 0$

So K is hermitian provided this condition is satisfied.

The condition is satisfied if they have the same value at the limits or if they vanish at the limits.

This condition is met by Quantum Mechanics functions which define the infinite dimension Hilbert space.

Hilbert Space and **Eigenvalues of K** and **Eigenvectors of K**

$K|k\rangle = k|k\rangle \Rightarrow \langle x|K|k\rangle = k\langle x|k\rangle = k\,\psi_k(x)$ ← eigenvector of K is a function

$$\int dx' \underbrace{\langle x|K|x'\rangle}_{-i\delta'(x-x')}\underbrace{\langle x'|k\rangle}_{\psi_k(x')} = -i\frac{d\psi_k}{dx} = k\,\psi_k(x) \Rightarrow \frac{d\psi_k}{\psi_k} = ik\,dx \Rightarrow \underbrace{\psi_k = Ae^{ikx}}_{\text{plane wave}}$$

Normalization: $1 = \langle k|k'\rangle = \int dx \langle k|x\rangle\langle x|k'\rangle = \int dx\, \psi_k^{*}\psi = |A|^2\int e^{i(k'-k)x}dx$

$1 = |A|^2 2\pi\,\delta(k-k') \Rightarrow \psi_k = \frac{1}{\sqrt{2\pi}}e^{ikx}$ eigenfunctions are component of eigenvector along that basis.

k is real
no other restrictions

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(5) Hilbert Space Coordinate vs Momentum Representation

x = coordinate
$\hbar k$ = momentum

$f(x) = \langle x|f\rangle$
$\tilde{f}(k) = \langle k|f\rangle$ } related but not the same

$\Rightarrow \frac{1}{2\pi}\int e^{ik(x-x')}dk = \delta(x-x')$

Relate By Fourier Transform

$\langle x|f\rangle = f(x) = \int dk \langle x|k\rangle\langle k|f\rangle = \int dk\, \Psi_k(x)\tilde{f}(k)$

$= \frac{1}{\sqrt{2\pi}}\int dk\, \underbrace{e^{ikx}}_{\text{eigenfunction of momentum op in } x}\tilde{f}(k)$

$\langle k|f\rangle = \tilde{f}(k) = \int dx \langle k|x\rangle\langle x|f\rangle = \int dx \frac{1}{\sqrt{2\pi}}e^{-ikx}f(x)$

Normalization

Previously showed that Hilbert Space allows only certain, complex functions

1) $\langle x|f\rangle = f(x)$, $\langle x|g\rangle = g(x)$ } only if $f^*(x)g(x)\Big|_a^b = 0$ are allowed

2) Normalizable $(-\infty, +\infty)$ have to vanish

3) Eigenfunctions of plane wave: $f(x) = e^{ikx}$ $g(x) = e^{ik'x}$
meets reqts:
$f^*g\Big|_{-\infty}^{\infty} = e^{i(k'-k)x}\Big|_{-\infty}^{+\infty}$

$\lim_{x\to\infty} e^{i(k'-k)x} = \lim_{\substack{L\to\infty \\ \Delta\to 0}} \int_L^{L+\Delta} e^{i(k'-k)x}dx = \frac{1}{\Delta}\left.\frac{e^{i(k'-k)x}}{i(k'-k)}\right|_L^{L+\Delta}$

$= \frac{1}{i(k'-k)\Delta}\left[e^{i(k'-k)(L+\Delta)} - e^{i(k'-k)L}\right] = \frac{e^{i(k'-k)L}}{i(k'-k)\Delta}\left[1 + i(k'-k)\Delta - 1\right]$

$= e^{i(k'-k)L}$

Put a Particle in a Box (1D)

$\Psi_k(x) = \frac{1}{\sqrt{L}}e^{ikx}$

Boundary condition
$e^{ikL} = 1 \Rightarrow kL = n\,2\pi \Rightarrow k_n = n\left(\frac{2\pi}{L}\right)$ count the states

$\therefore \frac{1}{L}\int_0^L \Psi_{k'}^*\Psi_k\, dx = \delta_{k'k} = \delta_{n'n}$ can avoid non-normalizable wavefunctions

Momentum

Previously: $K = iD$ is Hermetian in Hilbert space $\langle x|K|k\rangle = k\langle x|k\rangle = (\Psi_k(x))k$

$\int \langle x|K|x'\rangle\langle x'|k\rangle dx' = \int -i\delta'(x-x')\,\Psi_k(x')dx' = k\Psi_k(x) = -i\frac{d\Psi_k}{dx} \Rightarrow \Psi_k(x) = Ce^{ikx}$

Can show $C = \frac{1}{\sqrt{2\pi}}$ within a phase factor

$P_x = \hbar K$
$p_x = \hbar k$

$\langle p_x|p_x'\rangle = \delta(p_x - p_x') = \delta(\hbar(k-k')) = \frac{1}{\hbar}\delta(k-k')$

with this normalization eigenfunction correspond to eigenvector P_x

$\boxed{\Psi_k(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ip_x x/\hbar}}$

NOTE BOOK

Two Sets of Basis Vectors

$|x\rangle$ abstract coord space wavefunction $\Leftarrow$ fourier transform $\Rightarrow$ $|k\rangle$ abstract momentum space wavefth.

Consider an arbitrary ket vector $|f\rangle$

$$\langle x|f\rangle = f(x) = \frac{1}{\sqrt{2\pi}}\int \tilde{f}(k)e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int dk\, e^{ikx}\int f(x')e^{-ikx'}dx'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')dx' \int e^{ik(x-x')}dk$$

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x-x')dx' \quad \delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk = \frac{1}{2\pi}\left.\frac{e^{ik(x-x')}}{i(x-x')}\right|_{k=-\infty}^{k=+\infty} = 0 \text{ if } x \neq x'$$

$$\langle k|f\rangle = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int f(x)e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\int f(x')e^{-ikx'}dx'$$

X - basis Matrix Elements

$$\boxed{\langle x|X|x'\rangle = x'\langle x|x'\rangle = x'\delta(x-x') = x\delta(x-x')}$$

$$\boxed{\langle x|K|x'\rangle = -i\delta'(x-x')}$$

x, k are conjugate operators

$xk \neq kx \quad (xk-kx)|f\rangle \neq 0 = i\mathbb{I}|f\rangle$

$[X, K] = i\mathbb{I}$

If this is true for every component, then it is true for abstract k-vector
easy to prove, take components

K - basis (momentum) Matrix El.

$$\boxed{\langle k|K|k'\rangle = k'\langle k|k'\rangle = k'\delta(k-k') = k\delta(k-k')}$$

$$\langle k|X|k'\rangle = \iint dx\,dx' \langle k|x\rangle\langle x|X|x'\rangle\langle x'|k'\rangle$$

$$= \iint dx\,dx'\, \psi_k^*(x)[x\,\delta(x-x')]\psi_{k'}(x')$$

$$= \iint dx\,dx' \frac{1}{\cancel{2\pi}} e^{ik(x'-x)}\, x\,\delta(x-x')$$

write $xe^{i(k'-k)x} = i\frac{d}{dk}e^{i(k'-k)x}$

$$\langle k|X|k'\rangle = i\frac{d}{dk}\frac{1}{2\pi}\int e^{i(k'-k)}dx = i\frac{d}{dk}\delta(k-k')$$

$$\boxed{\langle k|X|k'\rangle = i\delta'(k-k')}$$

(A) $\langle x|XK|f\rangle = x\langle x|K|f\rangle = x\int dx'\langle x|K|x'\rangle\langle x'|f\rangle = x\int dx'(-i\delta'(x-x'))f(x') = -ix\frac{df}{dx}$

(B) $\langle x|KX|f\rangle = \int dx'\langle x|K|x'\rangle\langle x'|X|f\rangle = \int dx'(-i\delta'(x-x'))\underbrace{x'f(x')}_{g(x')} = -i\frac{d}{dx}g(x) = -i\frac{d}{dx}(x f(x))$

$\langle x|KX|f\rangle = -ix\frac{df}{dx} - if(x)$

$\therefore \langle x|XK - KX|f\rangle = \overset{(A)-(B)}{+if(x)} = i\langle x|\mathbb{I}|f\rangle$ QED

Heisenberg's Uncertainty Principle

$p_x = \hbar k \quad [X, K] = i\mathbb{I}\hbar$

State of System is defined by vector in Hilbert Space
Probability Amplitude: of finding a state g if is in state f : $\langle g|f\rangle$ complex scalar

observable k eigenvalue k: $\langle k|f\rangle$

$\langle x|f\rangle = f(x) \quad \langle f|f\rangle = 1 = \int\langle f|x\rangle\langle x|f\rangle dx = \int f^* f\,dx = \int|f(x)|^2 dx \Rightarrow P(x)dx = |f(x)|^2 dx$

$\langle k|f\rangle = \tilde{f}(k) \quad \int\langle f|k\rangle\langle k|f\rangle dk = \int\tilde{f}^*(k)\tilde{f}(k)dk \Rightarrow P(k)dk = |\tilde{f}(k)|^2 dk$ prob density in k-space

(6) Classical Mechanics: $F = ma = m\frac{d^2x}{dt^2}$ $\quad L = L(q_i, \dot{q}_i)$ $\quad S = \int_{t_i}^{t_f} L\,dt = \int_{q_{1i}\text{ at } t_1}^{q_{2i}\text{ at } t_2} L\,dt$

Lagrangian action

The action:
S will be a minimum for the allowed path in classical mechanics

action = momentum · length
= energy · time

$$S = \int_{t_1}^{t_2} L(q_{i1}, q_{i2})\,dt$$

$$\delta S = \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q_i}\right)\delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta \dot{q}_i\right]dt = 0 = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i\,dt$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

$\therefore$ $\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0}$

Newton's 2ND Law
only the path that gives you the least action is allowed if it is a conservative force

also define $p_i = \frac{\partial L}{\partial \dot{q}_i}$ conj. mom.

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \Leftrightarrow 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \Leftrightarrow \frac{dm\dot{x}}{dt} = \frac{\partial V}{\partial x} = F_x$

$H = \sum p_i \dot{q}_i - \mathcal{L}$

in QMech, there is probability amplitude
no single path, $\therefore$ path integral formulation

from Lagrangian we can prove: $\quad e^{iS/\hbar} \Rightarrow \sum(\ldots)e^{iS/\hbar}$ = only path w/ least action allowed

$\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\quad \dot{p}_i = -\frac{\partial H}{\partial \dot{q}_i}$ = generalized coordinates

Hamilton's Equations

Particle in an Electromagnetic Field

$\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$ in Gaussian units, E + B have the same units

Maxwell's Equations
$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ $\quad \vec{\nabla}\cdot\vec{B} = 0$
$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ $\quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

vacuum
$\vec{B} = \vec{\nabla}\times\vec{A} = 0$
so $\nabla\cdot\nabla\times A = 0$

Vector/Scalar Potential

$\nabla\times E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times A) = 0$

$\nabla\times\underbrace{\left(E + \frac{1}{c}\frac{\partial A}{\partial t}\right)}_{-\nabla\phi = \vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}} = 0$

$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$\frac{d\vec{p}}{dt} = \vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$

Lorentz

How to define Lagrangian

$\mathcal{L} = \frac{1}{2}m\vec{v}\cdot\vec{v} - q\phi + \frac{q}{c}\vec{v}\cdot\vec{A} = \frac{1}{2}m\sum \dot{x}_i\dot{x}_i - q\phi + \frac{q}{c}\sum \dot{x}_i A_i$

Canonically conjugate momentum

$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{q}{c}A_i = m\vec{v} + \frac{q}{c}\vec{A}$ $\quad m\vec{v} = \vec{p} - \frac{q}{c}\vec{A}$

$H = \sum p_i\dot{q}_i - \mathcal{L} = (m\vec{v} + \frac{q}{c}\vec{A})\cdot\vec{v} - \frac{1}{2}m(\vec{v})^2 + q\phi - \frac{q}{c}\vec{v}\cdot\vec{A} = \frac{1}{2}m\vec{v}\cdot\vec{v} + q\phi$
$= T + V$

$H = \frac{(\vec{p} - \frac{q}{c}\vec{A})^2}{2m} + q\phi$ in presence of electromagnetic field.

Poisson Brackets

$\dot{q} = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ $\omega(q_i, p_i)$: $\frac{d\omega}{dt} = \sum_i \left(\frac{\partial \omega}{\partial q_i}\dot{q}_i + \frac{\partial \omega}{\partial p_i}\dot{p}_i\right) = \sum\left(\frac{\partial \omega}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$

$$\frac{d\omega}{dt} = \{\omega, H\}_{P.B.}$$

$\{q_i, p_j\}_{PB} = \delta_{ij}$ replace Poisson Bracket with $\frac{1}{\hbar}$[commutator]

to get Quantum mechanics

$\{q_i, q_j\}_{PB} = 0$ $\{p_i, p_j\} = 0$ $\{q_i, p_j\}_{PB} = \delta_{ij}$

Two-Body Problem

$H = T + V = \frac{1}{2}m_1(\dot{\vec{r}}_1)^2 + \frac{1}{2}m_2(\dot{\vec{r}}_2)^2 + V(|\vec{r}_1 - \vec{r}_2|)$ $\vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$ $(m_1+m_2)\vec{R}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 = m_1\vec{r}_1 - m_2\vec{r}_1 + m_2\vec{r}_2$

$\Rightarrow \vec{r}_2 = \vec{R}_{cm} - \left(\frac{m_1}{m_1+m_2}\right)\vec{r}$ $\vec{r}_1 = \vec{R}_{cm} + \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$

$$H = \frac{1}{2}(m_1)\left(\dot{\vec{R}}_{cm} + \frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}(m_2)\left(\dot{\vec{R}}_{cm} - \frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2 + V(\vec{r})$$

$$\boxed{H = \frac{1}{2}M\dot{\vec{R}}_{cm}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 + V(\vec{r})}$$

$\mathcal{L} = T - V = \frac{1}{2}M(\dot{\vec{R}}_{cm})^2 + \frac{1}{2}\mu(\dot{\vec{r}})^2 - V(\vec{r})$ generalized coordinates $\vec{R}_{cm}, \vec{r}$

$$\frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{R}_{cm}}\right)}_{\text{Conjugate momentum}} - \underbrace{\frac{\partial L}{\partial R_{cm}}}_{=0} = 0$$

$\frac{dP_{cm}}{dt} = 0$ the center of mass momentum is conserved

when $P_{cm} = 0$ that is the center of mass frame

positronium (e^-, e^+) $\mu = \frac{m_e}{2}$ hydrogen $m_p \sim 2000\, m_e$ $\mu \sim m_e$