Quantum Mechanics   May 12, 2009   Meg Noah

9/10

19.4.1 Derive the inequality 19.4.44.

In the Born approximation $\psi_k = e^{i\vec{k}\cdot\vec{r}} + \psi_{sc} \Rightarrow e^{i\vec{k}\cdot\vec{r}}$ if $|\psi_{sc}| \ll |e^{i\vec{k}_i\cdot\vec{r}}|$ in the region $|\vec{r}'| \leq r_0$, and ψ_{sc} is largest near origen of $V(r)$.

To get the inequality start by calculating $\psi_{sc}(r=0)$ and $\psi_{inc}(r=0)$ and take the ratio (and $V(\vec{r}) \to V(r)$):

$$\left|\frac{\psi_{sc}(0)}{\psi_{inc}(0)}\right| = \left|\frac{\psi_{sc}(0)}{e^0}\right| = \frac{2\mu}{4\pi\hbar^2}\left|\int_{V'} \frac{e^{ikr'}}{r'} V(r)\, e^{-i\vec{k}_i\cdot\vec{r}'} d^3\vec{r}'\right|$$

The integral over the volume has a trival component $\int d\phi = 2\pi$ so that only $r^2 \sin\theta\, dr\, d\theta$ are left.

$$\left|\frac{\psi_{sc}(0)}{\psi_{inc}(0)}\right| = \left(\frac{2\mu}{4\pi\hbar^2}\right)2\pi\left|\iint \frac{e^{ikr'}}{r'} V(r') e^{-ik_i r'\cos\theta'} r'^2 \sin\theta'\, dr'\, d\theta'\right|$$

$$\left|\frac{\psi_{sc}(0)}{\psi_{inc}(0)}\right| = \frac{\mu}{\hbar^2}\left|\int\int_0^{\pi} e^{ikr'} r' V(r') e^{-ik_i r'\cos\theta}\, d\cos\theta\, dr'\right|$$

$$\left|\frac{\psi_{sc}(0)}{\psi_{inc}(0)}\right| = \frac{\mu}{\hbar^2}\left|\int e^{ikr'} \frac{r' V(r') e^{-ik_i r'(\cos\theta')}}{(-ik_i r')}\Bigg|_0^{\pi} dr'\right| = \frac{\mu 2}{\hbar^2 k_i}\left|\int \frac{e^{-ik_i r'} e^{ikr'} V(r') dr'}{(-i)}\right|$$

$$\therefore \left|\frac{\psi_{sc}(0)}{\psi_{inc}(0)}\right| = \frac{2\mu}{\hbar^2 k_i}\left|\int e^{ikr'} V(r') \left(\frac{e^{ikr'} - e^{ikr'}}{+i}\right)\right| \ll 1$$

$$\therefore \frac{2\mu}{\hbar^2 k}\left|\int e^{ikr'} V(r') \sin kr'\, dr'\right| \ll 1$$

19.5.1 Show that for a 100 MeV (kinetic energy) neutron incident on a fixed nucleus, $l_{max} \sim 2$.

$l_{max} \sim kr_0$ $\quad r_0 \sim 10^{-5}$ Å ~ 1 Fermi $\quad E = 100$ MeV $\quad \mu = 938$ MeV

$\hbar c \approx 200$ MeV Fermi

$$k = \frac{\sqrt{2E\mu}}{\hbar} = \frac{\sqrt{2E\mu c^2}}{\hbar c}$$

$$k = \frac{\sqrt{2(100\text{ MeV})(938\text{ MeV})}}{200\text{ MeV Fermi}} = \frac{433\text{ MeV}}{200\text{ MeV Fermi}} = 2.17\text{ Fermi}^{-1}$$

$l_{max} \sim kr_0 = (2.17\text{ Fermi}^{-1})(1\text{ Fermi}) = 2.17 \therefore l_{max} \sim 2$

NOTE BOOK

19.5.2 Derive 19.5.8 and provide the missing steps leading to the optical theorem 19.5.21.

This is based on Section 15.4 in Schaum's Outline:

$V(\vec{r}) = V(r) \leftarrow$ spherically symmetric, so we know the stationary wave function $\phi_k(r,\theta)$ and the scattering amplitude $f_k(\theta)$ can be expanded in terms of the Legendre polynomials ($P_\ell(\cos\theta)$)

$$\phi_k(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \frac{\chi_\ell(r) P_\ell(\cos\theta)}{r} \quad \text{and} \quad f_k(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell P_\ell(\cos\theta)$$

where A_ℓ, f_ℓ, and $\chi_\ell(r)$ are unknowns for for a particular $V(r)$. $\chi_\ell(r)$ satisfies the radial Schrödinger equation: $\left(R_\ell(r) = \frac{U_\ell(r)}{r}\right)$

$$\left[\frac{\partial^2}{\partial r^2} + k^2 - M(r) - \frac{\ell(\ell+1)}{r^2}\right]\chi_\ell(r) = 0 \quad \text{and} \quad \chi_\ell(0) = 0 \text{ and}$$

and also

$$\lim_{r\to\infty} \chi_\ell(r) \sim [A_\ell \underset{\text{Bessel}}{j_\ell(kr)} + B_\ell \underset{\text{Neumann}}{n_\ell(kr)}]r = \frac{C_\ell}{k}\sin(kr - \frac{\pi\ell}{2} + \underset{\text{phase shift}}{\delta_\ell})$$

$\lim_{r\to\infty} \chi_\ell^0(r) \sim \frac{C_\ell}{k}\sin(kr - \frac{\pi\ell}{2})$. The plane waves ~~are~~ can be expanded in terms of the Legendre polynomials $e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^\ell(2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$

Substitute e^{ikz}, $f_k(\theta)$, and $\phi_k(r,\theta)$ into the ~~expan~~ Schrödinger eq to obtain:

$$A_\ell = (2\ell+1) i^\ell e^{i\delta_\ell} \quad \text{and} \quad f_k(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2i\delta_\ell} - 1)P_\ell(\cos\theta)$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)\right|^2 \Rightarrow \sigma_T = \int |f|^2 d\Omega = \frac{4\pi}{k^2}\sum_{\ell=0}(2\ell+1)\sin^2\delta_\ell$$

since $j_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \frac{\ell\pi}{2})}{kr}$

(euler relation) $= \frac{e^{i(kr - \ell\pi/2)} - e^{-i(kr - \ell\pi/2)}}{ikr}$

$$e^{ikz} \xrightarrow[r\to\infty]{} \frac{1}{2ik}\sum_{\ell=0} i^\ell (2\ell+1)\left(\frac{e^{+i(kr-\ell\pi/2)} - e^{-i(kr - \frac{\ell\pi}{2})}}{r}\right)P_\ell(\cos\theta)$$

$$= \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[\frac{e^{ikr}}{r} - \frac{e^{-ikr-\ell\pi}}{r}\right]P_\ell(\cos\theta)$$

12.5.8

$$f_k(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2i\delta_\ell} - 1)P_\ell(\cos\theta) \Rightarrow f_k(0) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2i\delta_\ell} - 1)$$

$$f_k(0) = \frac{i}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

$$\text{Im}(f_k(0)) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

$$\sigma \int |f|^2 d\Omega = \frac{4\pi}{k^2}\sum(2\ell+1)\sin^2\delta_\ell = \frac{4\pi}{k}\text{Im}(f_k(0))$$

Quantum Mechanics Meg Noah

19.5.3 (1) Show that $\sigma_0 \to 4\pi r_0^2$ for a hard sphere as $k \to 0$.

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell \qquad 19.5.31$$

$$\lim_{k\to 0} \tan\delta_\ell \sim \delta_\ell \propto (kr_0)^{2\ell+1} \qquad 19.5.29$$

$$\therefore \lim_{k\to 0} \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\left[(kr_0)^{2\ell+1}\right]^2$$

$$\therefore \lim_{k\to 0} \sigma_0 = \frac{4\pi}{k^2}(1)(kr_0)^2 = 4\pi r_0^2$$

(2) Consider the other extreme of kr_0 very large. From 19.5.27 and the asymptotic forms of j_ℓ and n_ℓ show that:

$$\sin^2\delta_\ell \xrightarrow[kr_0\to\infty]{} \sin^2(kr_0 - \ell\pi/2)$$

so that

$$\sigma = \sum_{\ell=0}^{\ell_{max}-kr_0} \sigma_\ell \sim \frac{4\pi}{k^2}\int_0^{kr_0}(2\ell)\sin^2\delta_\ell \, d\ell \sim 2\pi r_0^2$$

if we approx sum over ℓ by an integral $2\ell+1$ by 2ℓ and the oscillating function $\sin^2\delta$ by its average value of 1/2.

$$\delta_\ell = \tan^{-1}\left[\frac{j_\ell(kr_0)}{n_\ell(kr_0)}\right] \xrightarrow[kr_0\to\infty]{} \tan^{-1}\left[\frac{\sin(kr_0-\ell\pi/2)/kr_0}{-\cos(kr_0-\ell\pi/2)/kr_0}\right] = -kr_0 + \ell\pi/2$$

$$\therefore \sin^2\delta_\ell \to \sin^2(kr_0 - \ell\pi/2)$$

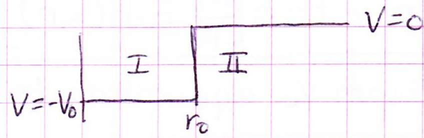
19.5.4 Show that the S-wave phase shift for a square well of depth V_0 and range r_0 is

$$\delta_0 = -kr_0 + \tan^{-1}\left[\frac{k}{k'}\tan(k'r_0)\right]$$

where k' and k are the wavenumbers inside and outside the well.

(This was pretty much done in class)

$r<r_0 \quad V(r) = -V_0$
$r>r_0 \quad V(r) = 0$



E>0 so scattering

The Schrödinger equation for $r<r_0$ (I):

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi_I - V_0\Psi_I = E\Psi_I \Rightarrow \nabla^2\Psi_I + k_I^2\Psi_I = 0 \text{ where } k_I^2 = \frac{2\mu}{\hbar^2}(E+V_0)$$

$$\therefore \Psi_I = R_{n\ell}Y_\ell^m \Rightarrow A_\ell j_\ell(k_I r) = R_{n\ell}$$ (note: can only have Bessel ftn because valid at $r=0$)

now for $r > r_0$ the Schrödinger equation is:

$$(\nabla^2 + k^2)\Psi_{II} = 0 \qquad k_{II}^2 = \frac{2\mu E}{\hbar^2}$$

$$R_{II}(r) = A_\ell'\left[j_\ell + \frac{B_\ell'}{A_\ell'}\eta_\ell(k_2 r_2)\right] = A_\ell'\left[j_\ell(k_{II} r_{II}) - \tan\delta_\ell\,\eta_\ell(k_{II} r_{II})\right]$$

S-wave, $l=0$ a better approach is to use $U(r)/r = R(r)$ and

$$\frac{d^2U}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V_0 - \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right]U = 0 \qquad \text{S-wave, } l=0$$

$U=0$ at $r=0$

$$\Rightarrow \frac{d^2U_I}{dr^2} + \frac{2\mu}{\hbar^2}(E+V_0)U_I = 0 \Rightarrow U_I = A\sin(k_1 r) + \cancel{B\cos(k r)} \qquad k_1^2 = \frac{2\mu(E+V_0)}{\hbar^2}$$

also

$$\Rightarrow \frac{d^2U_{II}}{dr^2} + \frac{2\mu}{\hbar^2}E\,U_{II} = 0 \Rightarrow U_{II} = B\sin(k_2 r) + C\cos(k_2 r) = B\left[\sin(k_2 r) + \frac{C}{B}\cos(k_2 r)\right]$$

$$= B\left[\sin(k_2 r) - \tan\delta_0 \cos(k_2 r)\right]$$

$$= \frac{B}{\cos\delta_0}\left[\cos\delta_0 \sin k_2 r - \sin\delta_0 \cos k_2 r\right] = \frac{B}{\cos\delta_0}\sin(k_2 r - \delta_0)$$

at $r = r_0$:

BC#1 $\quad U_I = U_{II} = \dfrac{B}{\cos\delta_0}\sin(k r_0 - \delta_0)$

BC#2 $\quad \dfrac{1}{U_I}U_I' = \dfrac{1}{U_{II}}U_{II}'$

$$\frac{1}{A\sin(k_1 r)}Ak_1\cos(k_1 r) = \frac{Ck_2}{C\sin(k_2 r - \delta_0)}\cos(k_2 r - \delta_0)$$

$$\frac{k_1}{\tan(k_1 r)} = \frac{k_2}{\tan(k_2 r - \delta_0)}$$

$$\tan(k_2 r - \delta_0) = \frac{k_2}{k_1}\tan(k_1 r)$$

$$\boxed{\delta_0 = -k_2 r + \tan^{-1}\left[\frac{k_2}{k_1}\tan(k_1 r)\right]}$$

Changing depth V_0 equivalent to changing k_1.

If $k_1 \sim k_{1n} = (2n+1)\pi/2r_0 \quad \delta_0 = \delta_b^{\,0} + \tan^{-1}\left(\dfrac{\Gamma/2}{E_0 - E}\right) = \tan^{-1}\left(\dfrac{\hbar^2 k_n}{\mu r_0(E_0 - E)}\right)$

$E_0 = \dfrac{\hbar^2 k_0^2}{2\mu} \qquad \tan^{-1}\dfrac{\hbar^2 k_n}{\mu r_0(E_0 - E)} = \tan^{-1}\left(\dfrac{2\mu E}{k r_0(E_0 - E)}\right) = \tan^{-1}\dfrac{2r_0\, 2E}{(2n+1)\pi r_0(E_0 - E)} =$

19.5.6 (Optical Theorem)

(1) Show that the radial component of the current density due to interference between the incident + scattered waves is

$$j_r^{int} \underset{r\to\infty}{\sim} \left(\frac{\hbar k}{\mu}\right)\left(\frac{1}{r}\right) \mathrm{Im}\left\{ i e^{ikr(\cos\theta - 1)} f^*(\theta)\cos\theta + i e^{ikr(1-\cos\theta)} f(\theta) \right\}$$

from Gasiorowicz

Solution to $V=0$ Schrodinger eq. is $\psi = e^{i\vec{k}\cdot\vec{r}}$ implies

$$\vec{J} = \frac{\hbar}{i2\mu}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = \frac{\hbar\vec{k}}{\mu} \qquad \vec{k} = k\hat{z}$$

$$e^{i\vec{k}\cdot\vec{r}} \xrightarrow[r\to\infty]{} \frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)i^l\left[\frac{e^{-i(kr - l\pi/2)}}{r} - \frac{e^{+i(kr - l\pi/2)}}{r}\right]P_l(\cos\theta)$$

Conservation of particles puts condition on presence of ~~sy~~ spherically symmetric potential such that:

$$\psi(\vec{r}) \to \frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)i^l\left[\frac{e^{-i(kr-l\pi/2)}}{r} - S_l(k)\frac{e^{i(kr-l\pi/2)}}{r}\right]P_l(\cos\theta)$$

where $|S_l(k)| = 1$

$$\therefore \psi(\vec{r}) \Rightarrow e^{i\vec{k}\cdot\vec{r}} + \left[\sum_{l=0}^{\infty}(2l+1)\frac{[S_l(k)-1]}{2ik}P_l(\cos\theta)\right]\frac{e^{ikr}}{r}$$

$$\vec{j} = \frac{\hbar}{2im}\left\{\left[e^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{ikr}}{r}\right]^*\nabla\left[e^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{ikr}}{r}\right] - \left[e^{-\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{-i\vec{k}\cdot\vec{r}}}{r}\right]^*\nabla\left[e^{-i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{-i\vec{k}\cdot\vec{r}}}{r}\right]\right\}$$

where $f(\theta) = \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta) \qquad f_l(k) = [S_l(k)-1]/2ik$

Calculating the gradient: (setting $1/r^3$ terms $\to 0$ since $r\to\infty$

$$\vec{j} = \frac{\hbar}{2i\mu}\left\{\left[e^{-i\vec{k}\cdot\vec{r}} + f^*(\theta)\frac{e^{-ikr}}{r}\right]\left[i\vec{k}\, e^{i\vec{k}\cdot\vec{r}} + \hat{\theta}\frac{1}{r}\frac{\partial f(\theta)}{\partial\theta}\frac{e^{ikr}}{r} + \hat{r} f(\theta)\left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right)\right] - \text{complex conjugate}\right\}$$

$$\vec{j} = \frac{\hbar}{2i\mu}\left[i\vec{k} + i\vec{k}f^*(\theta)\frac{e^{-ikr(1-\cos\theta)}}{r} + ik\hat{r}f(\theta)\frac{e^{ikr(1-\cos\theta)}}{r} + ik\hat{r}|f(\theta)|^2\frac{1}{r^2} - \hat{r}f(\theta)\frac{e^{ikr(1-\cos\theta)}}{r^2} + \hat{\theta}\frac{\partial f(\theta)}{\partial\theta}\frac{e^{ikr(1-\cos\theta)}}{r^2} - \text{complex conjugate}\right\}$$

$$\vec{j} = \frac{\hbar \vec{k}}{\mu} + \frac{\hbar k}{\mu}\hat{r}\,|f(\theta)|^2\frac{1}{r^2} + \frac{\hbar \vec{k}}{2\mu}\frac{1}{r}\left[f^*(\theta)e^{-ikr(1-\cos\theta)} + f(\theta)e^{ikr(1-\cos\theta)}\right]$$

$$+\frac{\hbar k}{2\mu}\frac{\hat{r}}{r}\left[f^*(\theta)e^{-ikr(1-\cos\theta)} + f(\theta)e^{ikr(1-\cos\theta)}\right]$$

$$-\frac{\hbar}{2i\mu}\frac{\hat{r}}{r^2}\left[f(\theta)\,e^{ikr(1-\cos\theta)} - f^*(\theta)e^{-ikr(1-\cos\theta)}\right]$$

$$+\frac{\hbar}{2i\mu}\frac{\hat{\theta}}{r^2}\left[\frac{\partial f(\theta)}{\partial\theta}\,e^{ikr(1-\cos\theta)} - \frac{\partial f^*(\theta)}{\partial\theta}\,e^{-ikr(1-\cos\theta)}\right]$$

now just want the radial component:

$$\vec{j}_r = \vec{j}\cdot\hat{r} = \frac{\hbar k\cos\theta}{\mu} + \frac{\hbar k}{\mu}|f(\theta)|^2\frac{1}{r^2} + \frac{\hbar k\cos\theta}{2\mu}\frac{1}{r}\left[f^* e^{-ikr(1-\cos\theta)} + f e^{ikr(1-\cos\theta)}\right]$$

$$+\frac{\hbar k}{2\mu r}\left[f^* e^{-ikr(1-\cos\theta)} + f e^{+ikr(1-\cos\theta)}\right]$$

$$-\frac{\hbar}{2i\mu r^2}\left[f\, e^{ikr(1-\cos\theta)} - f^* e^{-ikr(1-\cos\theta)}\right]$$

for $r\to\infty$ set 2nd & last term to zero.

~~we had the term $f_\ell(k) = [S_\ell(k)-1]/2ik$ in $f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta)$ so the $S_\ell(k)$ is set to remove waves traveling in the $e^{ikr(1-\cos\theta)}$ direction $(-z)$~~

finally the $\hbar k\cos\theta/\mu$ term is negligable compared to the other terms so that

$$\vec{j}_r = \frac{\hbar k}{\mu}\frac{1}{r}\,\mathrm{Im}\left[i e^{ikr(\cos\theta-1)}f^*(\theta)\cos\theta + i e^{ikr(1-\cos\theta)}f(\theta)\right]$$

(2) As long as $\theta \neq 0$ $\langle \vec{j}^{\,int}\rangle$ over a small solid angle is zero because $r\to\infty$

a small solid angle goes as $1/r^2$ $\vec{j}$ has $1/r$ so $1/r^3 \to \infty$ because $r\to\infty$

(3) Integrate $\vec{j}_r$ over a tiny cone in the forward direction

$$\int_{\text{forward cone}} j_r^{int}\, r^2\, d\Omega = \int_0^{2\pi}\int_0^{\theta} j_r\, r^2\, d\cos\theta\, d\phi = 2\pi \cancel{\int_0^{\delta\theta} j_r r^2 d\cos\theta} \quad 2\pi r^2 \int_0^{\delta\theta} j_r\, d(\cos\theta)$$

$$\cancel{\lim_{\delta\theta\to 0} 2\pi r^2\int_0^{\delta\theta} j_r\, d(\cos\theta) = -2\pi r^2\, j_r(r=r,\cos\theta=1) = -2\pi r^2\frac{\hbar k}{\mu}\frac{1}{r}\left[2f(\theta)\right] = -\frac{4\pi\hbar k}{\mu}f(\theta)}$$

$$\int = 2\pi r^2\int_{\delta}^{\delta}\left[\frac{\hbar k}{\mu}\frac{1}{r}\right]\mathrm{Im}\left\{i e^{ikr(\cos\theta-1)}f^*\cos\theta + i e^{ikr(1-\cos\theta)}f(\theta)\right\}d\cos\theta \Longrightarrow -\left(\frac{\hbar k}{\mu}\right)\frac{4\pi}{k}\,\mathrm{Im}[f(0)]$$

algebra & calculus & magic